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GENERAL SOLUTION OF QUANTUM MASTER EQUATION IN FINITE–DIMENSIONAL CASE.

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Abstract

The general solution to the quantum master equation (and its $Sp(2)$ symmetric counterpart) is constructed explicitly in case of finite number of variables. It is shown that the finite–dimensional solution is physically trivial and thus can not be extended directly to cover the case of a local field theory. In this way we conclude that the locality condition plays an important role by making it possible to obtain nontrivial physical results when quantizing gauge field theories on the basis of field–antifield formalism.

1 Introduction

When quantizing gauge theories directly within Lagrangian formalism [1] – [5] one usually proceeds by solving the quantum master equation

$$\Delta e^{\frac{i}{\hbar}S}=0 \tag{1.1}$$

or its $Sp(2)$ counterparts [6] – [9]

$$\bar{\Delta}^a e^{\frac{i}{\hbar}S}=0,\quad a=1,2, \tag{1.2}$$

where Δ and $\bar{\Delta}^a$ are Fermionic nilpotent operators:

$$\Delta^2=0,\quad \bar{\Delta}^a\bar{\Delta}^b+\bar{\Delta}^b\bar{\Delta}^a=0. \tag{1.3}$$

The operators Δ and $\bar{\Delta}^a$ are local second-order differential operators of the following characteristic structure:

$$\Delta=\int dx(-1)^{\varepsilon_A}\frac{\delta}{\delta\Phi^A(x)}\frac{\delta}{\delta\Phi^*_A(x)}. \tag{1.4}$$

When studying various properties of solutions to the quantum master equation (arbitrariness of a solution, gauge independence and reparametrization invariance of physical quantities, structure of renormalization) one considers the matter in the same manner as if one dealt with the theory of finite number of variables, in which the operator Δ were of the form

$$\Delta=\sum_{A=1}^{N}(-1)^{\varepsilon_A}\frac{\partial}{\partial\Phi^A}\frac{\partial}{\partial\Phi^*_A}. \tag{1.5}$$

However, when extending the results, obtained in a finite–dimensional case, to a local field theory, one should be very careful (for a review of some results on local gauge field theory see refs. [10], [11] and references therein). This is because of the fact that various transformations like a change of integration variables or an inversion of matrices (operators), which are sensible in a finite-dimensional case, may appear to be inadmissible when extending them to the case of a local field theory.

For example, in the case of a system with finite number of Bosonic variables, the actions

$$S=\frac{1}{2}\Phi^A\Lambda_{AB}\Phi^B \tag{1.6}$$

with a positive-defined matrix Λ_{AB} are equivalent (up to an improper change of variables) to the following canonical action

$$S=\frac{1}{2}\sum_A\Phi^A\Phi^A. \tag{1.7}$$

At the same time, within a local field theory, the actions

$$S_i=\frac{1}{2}\int dx(\partial_\mu\Phi^A\partial_\mu\Phi^A-m_i^2\Phi^A\Phi^A),\quad i=1,2, \tag{1.8}$$

are nonequivalent for $m_1\neq m_2$.

Another example is as follows. One should frequently deal with classes of functions determined modulo contributions vanishing at the extremals. In a theory with finite number of variables, governed by the action (1.6), the only representatives of the mentioned classes are constants. On the other hand, in a local field theory with the action (1.8), nontrivial classes certainly do exist with the representatives $\int dxF(\Phi,\partial_\mu\Phi)$.

In the present work we find the general solution to the equations

$$\Delta X=0\quad\text{and}\quad\bar{\Delta}^aX=0, \tag{1.9}$$

and also

$$\Delta e^{\frac{i}{\hbar}S}=0\quad\text{and}\quad\bar{\Delta}^a e^{\frac{i}{\hbar}S}=0, \tag{1.10}$$

for some operators Δ and $\bar{\Delta}^a$, in case of finite number of variables. From the field theory point of view, it appears in all the cases considered that the solutions to the equations (1.9) and (1.10) are physically trivial. Thus the results obtained, which are apparently interesting even by themselves, show actually the importance of the locality condition in quantum field theory. It becomes quite clear that the locality condition is, in fact, the only requirement which allows one to obtain physically nontrivial results within the formalism based on the quantum master equation.

All the equations under consideration are solved in terms of formal power series expansions with respect to all variables the functions X and S depend on, and also with respect to the Planck constant entering the S as well.

2 Equation $\Delta X = 0$.

Let us denote $\Phi^A = (u^i, \xi^\alpha)$, where $u^i, i = 1, \ldots, n_+$ and $\xi^\alpha, \alpha = 1, \ldots, n_-$ are Bosonic and Fermionic variables, respectively. In this notation the operator Δ takes the form

$$\Delta = (-1)^{\varepsilon_A} \frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^*_A} = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^*_i} - \frac{\partial}{\partial \xi^\alpha} \frac{\partial}{\partial \xi^*_\alpha}, \quad \Delta^2 = 0, \tag{2.1}$$

where the variables u^*_i (ξ^*_α) are Fermionic (Bosonic).

To construct the general solution to the equation

$$\Delta X = 0, \tag{2.2}$$

let us introduce the operator Γ,

$$\Gamma = (-1)^{\varepsilon_A} \Phi^A \Phi^*_A = u^i u^*_i - \xi^\alpha \xi^*_\alpha. \tag{2.3}$$

with the properties

$$\Gamma\Delta + \Delta\Gamma = N, \quad \Gamma^2 = 0, \quad [\Delta, N] = [\Gamma, N] = 0, \tag{2.4}$$

$$N = u^i \frac{\partial}{\partial u^i} + \xi^*_\alpha \frac{\partial}{\partial \xi^*_\alpha} + \left(n - u^*_i \frac{\partial}{\partial u^*_i} - \xi^\alpha \frac{\partial}{\partial \xi^\alpha} \right), \quad n = n_+ + n_-. \tag{2.5}$$

Let us apply the operator N to X, and then make use of the equation (2.2):

$$NX = \Delta\Gamma X. \tag{2.6}$$

If the operator N can be inverted, then the function X can be represented in the form

$$X = \Delta Y, \quad Y = \Gamma N^{-1} X. \tag{2.7}$$

Since every Fermionic variable can enter X at most linearly, it follows from the representation (2.5) for N that NX can vanish iff X does not depend on Bosonic variables, while the dependence on Fermionic variables comes only from their complete product. Thus the general solution to the equation (2.2) can be represented in the form:

$$X = \Delta Y + c \prod_i u^*_i \prod_\alpha \xi^\alpha, \tag{2.8}$$

where Y is an arbitrary function, and c is an arbitrary constant. It is of the most importance that the second term in r.h.s. in (2.8) cannot be represented in the form ΔZ. Indeed, otherwise Z would contain $n + 1$ Fermionic co–multipliers, which situation would be a contradiction. In other words one can say that the expression (2.8) describes the cohomology group of the operator Δ as acting on the space of formal power series expansions with respect to all variables. The representation (2.8) was independently obtained by K.Bering [12].

Let us assume, in analogy with field theory, that original set of variables Φ is split into two subsets: $\Phi = (\varphi, c)$, or $u = (\varphi_u, c_\xi)$, $\xi = (\varphi_\xi, c_u)$, and the following ghost number values are assigned to all the variables

$$\mathrm{gh}(\varphi_u) = \mathrm{gh}(\varphi_\xi) = 0, \quad \mathrm{gh}(c_\xi) = \mathrm{gh}(c_u) = 1, \quad \mathrm{gh}(\Phi^*) = -\mathrm{gh}(\Phi) - 1. \tag{2.9}$$

Then we have

$$\mathrm{gh}\left(\prod_i u_i^* \prod_\alpha \xi^\alpha\right) = n_{c_u} - n_{\varphi_u^*} - 2n_{c_\xi^*}. \tag{2.10}$$

In "physical" situation $n_{c_u} < n_{\varphi_u} = n_{\varphi_u^*}$, so that $\mathrm{gh}(\prod u_i^* \prod \xi^\alpha) < 0$, and thus, in the zero ghost number sector, the general solution to the equation (2.2) has the form

$$X = \Delta Y, \quad \mathrm{gh}(X) = 0, \tag{2.11}$$

which is physically trivial.

An analogous consideration shows that the general solution to the equation

$$\Delta_1 X = 0, \quad \Delta_1 \equiv \Gamma, \tag{2.12}$$

has the form

$$X = \Delta_1 Y + c \prod_i u_i^* \prod_\alpha \xi^\alpha. \tag{2.13}$$

3 Equation $\bar{\Delta}^a X = 0$.

Let $6n$ be a total number of variables which are, in their own turn, split into three subsets as follows

$$\begin{aligned}
&\Phi^A = (u^i, \xi^\alpha), \quad i = 1, \ldots, n_+, \quad \alpha = 1, \ldots, n_-, \quad n_+ + n_- = n,\\
&\bar{\Phi}_A = (\bar{u}_i, \bar{\xi}_\alpha),\\
&\Phi^*_{Aa} = (u^*_{ia}, \xi^*_{\alpha a}), \quad a = 1, 2,\\
&\varepsilon(\Phi^A) = \varepsilon(\bar{\Phi}_A) = \varepsilon_A, \quad \varepsilon(\Phi^*_{Aa}) = \varepsilon_A + 1.
\end{aligned} \tag{3.1}$$

Operators $\bar{\Delta}^a$ have the form

$$\bar{\Delta}^a = (-1)^{\varepsilon_A} \frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^*_{Aa}} + \varepsilon^{ac} \Phi^*_{Ac} \frac{\partial}{\partial \bar{\Phi}_A} = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^*_{ia}} - \frac{\partial}{\partial \xi^\alpha} \frac{\partial}{\partial \xi^*_{\alpha a}} + \varepsilon^{ac} u^*_{ic} \frac{\partial}{\partial \bar{u}_i} + \varepsilon^{ac} \xi^*_{\alpha c} \frac{\partial}{\partial \bar{\xi}_\alpha}, \tag{3.2}$$

$$\varepsilon^{ba} = -\varepsilon^{ab}, \quad \varepsilon^{12} = -1.$$

and, thus, are nilpotent

$$\bar{\Delta}^a \bar{\Delta}^b + \bar{\Delta}^b \bar{\Delta}^a = 0, \quad \varepsilon(\bar{\Delta}^a) = 1. \tag{3.3}$$

These operators determine the quantum master equation in $Sp(2)$ symmetric formulation of gauge theories.

To solve the equation

$$\bar{\Delta}^a X = 0 \tag{3.4}$$

let us introduce the operators $\Gamma_a(\theta)$,

$$\Gamma_a(\theta) = \theta^A \Gamma_{Aa}, \quad \Gamma_{Aa} = (-1)^{\varepsilon_A} (\Phi^*_{Aa} \Phi^A + \varepsilon_{ac} \bar{\Phi}_A \frac{\partial}{\partial \Phi^*_{Ac}}), \tag{3.5}$$

$$\varepsilon_{ab}\varepsilon^{bc} = \delta^c_a, \quad \varepsilon(\theta^A) = 0$$

(there is no summation over A when defining Γ_{Aa}). The operators $\Gamma_a(\theta)$ are nilpotent

$$\Gamma_a(\theta_1)\Gamma_b(\theta_2) + \Gamma_b(\theta_2)\Gamma_a(\theta_1) = 0. \tag{3.6}$$

Also, the following relations hold

$$\begin{aligned}
&\bar{\Delta}^a\Gamma_b(\theta) + \Gamma_b(\theta)\bar{\Delta}^a = \delta^a_b N(\theta), \\
&[N(\theta), \bar{\Delta}^a] = [N(\theta), \Gamma_a(\theta_1)] = [N(\theta), N(\theta_1)] = 0, \\
&N(\theta) = \theta^A N_A, \quad N_i = 1 + \hat{n}_{u^i} - \hat{n}_{u^*_{i1}} - \hat{n}_{u^*_{i2}} - \hat{n}_{\bar{u}_i}, \quad N_\alpha = 1 + \hat{n}_{\xi^*_{\alpha 1}} + \hat{n}_{\xi^*_{\alpha 2}} + \hat{n}_{\bar{\xi}_\alpha} - \hat{n}_{\xi^\alpha},
\end{aligned} \tag{3.7}$$

where $\hat{n}_{u^i} = u^i\partial/\partial u^i$ and so on. Let us apply the operator $N^2(\theta)$ to the function X, and then make use of the equation (3.4)

$$N^2(\theta)X = \bar{\Delta}^2\bar{\Delta}^1\Gamma_1(\theta)\Gamma_2(\theta)X. \tag{3.8}$$

If the operator $N^2(\theta)$ can be inverted for some choice of parameters θ^A, then X can be represented in the form

$$X = \bar{\Delta}^2\bar{\Delta}^1 Y = \frac{1}{2}\varepsilon_{ab}\bar{\Delta}^b\bar{\Delta}^a Y, \quad Y = \Gamma_1(\theta)\Gamma_2(\theta)N^{-2}(\theta)X. \tag{3.9}$$

The operator $N^2(\theta)$ is noninvertible if X satisfies the equations

$$N_A X = 0, \quad \forall A. \tag{3.10}$$

Thus the general solution to the equations (3.4) can be represented in the form

$$X = \bar{\Delta}^2\bar{\Delta}^1 Y + X_B Q, \quad Q \equiv \prod_\alpha \xi^\alpha, \tag{3.11}$$

where X_B depends only on the variables u^i, u^*_{ia}, $\bar{u}_i$, and satisfies the equations

$$\begin{aligned}
&\bar{\Delta}^a_B X_B = 0, \quad (1 + \hat{n}_{u^i})X_B = (\hat{n}_{u^*_{i1}} + \hat{n}_{u^*_{i2}} + \hat{n}_{\bar{u}_i})X_B, \\
&\bar{\Delta}^a_B = \Delta^a_B + d^a, \quad \Delta^a_B = \frac{\partial}{\partial u^i}\frac{\partial}{\partial u^*_{ia}}, \quad d^a = \varepsilon^{ac}u^*_{ic}\frac{\partial}{\partial \bar{u}_i},
\end{aligned} \tag{3.12}$$

(recall that $u^i, \bar{u}_i$ and u^*_{ia} are Bosonic and Fermionic variables, respectively). Let us expand X_B in power series in u^*_{ia}:

$$X_B = \sum_{k=0}^{2n_+} X^{(k)}, \quad (\sum_{i,a}\hat{n}_{u^*_{ia}})X^{(k)} = kX^{(k)}. \tag{3.13}$$

It follows that $X^{(0)}$ satisfies the equation

$$\Delta^a_B X^{(0)} = 0. \tag{3.14}$$

It is shown in Appendix that the general solution to the equation

$$\Delta^a_B Z = 0 \tag{3.15}$$

can be represented in the form

$$\begin{aligned}
&Z = \Delta^2_B\Delta^1_B Z' + \sum_{n_1+n_2 \le n_+} C_{i_1\ldots i_{n_1}j_1\ldots j_{n_2}}\frac{\partial}{\partial u^*_{i_1 1}}\cdots\frac{\partial}{\partial u^*_{i_{n_1}1}}Q_1\frac{\partial}{\partial u^*_{j_1 2}}\cdots\frac{\partial}{\partial u^*_{j_{n_2}2}}Q_2, \\
&\frac{\partial}{\partial u^*_{ia}}C_{i_1\ldots j_{n_2}} = \frac{\partial}{\partial u^i}C_{i_1\ldots j_{n_2}} = 0, \quad Q_1 \equiv \prod_i u^*_{i1}, \quad Q_2 \equiv \prod_i u^*_{i2},
\end{aligned} \tag{3.16}$$

where coefficients $C_{i_1\dots j_{n_2}}$ are antisymmetric under permutation of any two neighboring indices. Thus we have

$$X^{(0)} = \Delta^2_B\Delta^1_B Y^{(2)}, \tag{3.17}$$

and X_B rewrites in the form

$$\begin{aligned}
&X_B = \bar\Delta^2_B\bar\Delta^1_B Y^{(2)} + \sum_{k=1}^{2n_+} X_1^{(k)},\\
&\Delta^a_B X_1^{(1)} = 0, \quad X_1^{(1)} = \Delta^2_B\Delta^1_B Y^{(3)},\\
&X_B = \bar\Delta^2_B\bar\Delta^1_B (Y^{(2)} + Y^{(3)}) + \sum_{k=2}^{2n_+} X_2^{(k)}.
\end{aligned} \tag{3.18}$$

Continuing this process, we obtain

$$\begin{aligned}
&X_B = \bar\Delta^2_B\bar\Delta^1_B (\sum_{k=2}^{n_++1} Y^{(k)}) + \sum_{k=n_+}^{2n_+} X_{n_+}^{(k)},\\
&\Delta^a_B X_{n_+}^{(n_+)} = 0.
\end{aligned} \tag{3.19}$$

The general solution to the equation (3.19)) is the following

$$X_{n_+}^{(n_+)} = \Delta^2_B\Delta^1_B Y^{(n_++2)} + \sum_{m=0}^{n_+} a_m(\bar u)\varepsilon_{i_1\dots i_m i_{m+1}\dots i_{n_+}} \frac{\partial}{\partial u^*_{i_1 1}}\cdots\frac{\partial}{\partial u^*_{i_m 1}} Q_1 \frac{\partial}{\partial u^*_{i_{m+1} 2}}\cdots\frac{\partial}{\partial u^*_{i_{n_+} 2}} Q_2, \tag{3.20}$$

here $\varepsilon_{i_1\dots i_{n_+}}$ is a totally antisymmetric constant tensor, and $\varepsilon_{12\dots n_+} = 1$. The second equation in (3.12) yields

$$\frac{\partial}{\partial \bar u_i} a_m(\bar u) = 0, \quad a_m = \mathrm{const}. \tag{3.21}$$

Thus we get

$$\begin{aligned}
&X_B = \bar\Delta^2_B\bar\Delta^1_B (\sum_{k=2}^{n_++2} Y^{(k)}) +\\
&\quad \sum_{m=0}^{n_+} a_m\varepsilon_{i_1\dots i_m i_{m+1}\dots i_{n_+}} \frac{\partial}{\partial u^*_{i_1 1}}\cdots\frac{\partial}{\partial u^*_{i_m 1}} Q_1 \frac{\partial}{\partial u^*_{i_{m+1} 2}}\cdots\frac{\partial}{\partial u^*_{i_{n_+} 2}} Q_2 + \sum_{k=n_++1}^{2n_+} X_{n_++1}^{(k)},\\
&\Delta^a_B X_{n_++1}^{(n_++1)} = 0,\\
&X_{n_++1}^{(n_++1)} = \Delta^2_B\Delta^1_B Y^{(n_++2)} + \sum_{m=0}^{n_+-1} C_{i_1\dots i_m i_{m+1}\dots i_{n_+-1}} \frac{\partial}{\partial u^*_{i_1 1}}\cdots\frac{\partial}{\partial u^*_{i_m 1}} Q_1 \frac{\partial}{\partial u^*_{i_{m+1} 2}}\cdots\frac{\partial}{\partial u^*_{i_{n_+-1} 2}} Q_2.
\end{aligned} \tag{3.22}$$

Now, the second of the equations (3.12) yields

$$C_{i_1\dots i_{n_+-1}} = 0. \tag{3.23}$$

Continuing this process, we obtain finally

$$X_B = \bar{\Delta}^2_B \bar{\Delta}^1_B \tilde{Y} + \sum_{m=0}^{n_+} a_m F_m, \tag{3.24}$$

$$F_m = \varepsilon_{i_1 \ldots i_m i_{m+1} \ldots i_{n_+}} \frac{\partial}{\partial u^*_{i_1 1}} \cdots \frac{\partial}{\partial u^*_{i_m 1}} Q_1 \frac{\partial}{\partial u^*_{i_{m+1} 2}} \cdots \frac{\partial}{\partial u^*_{i_{n_+} 2}} Q_2.$$

Then we find for X

$$X = \bar{\Delta}^2 \bar{\Delta}^1 Y + \sum_{m=0}^{n_+} a_m F_m Q. \tag{3.25}$$

Let us show that the second term in (3.25) represents nontrivial "cohomologies" of the operators $\bar{\Delta}^a$ in the sense that there is no combination of functions $F_m Q$, which can be represented in the form $\bar{\Delta}^2 \bar{\Delta}^1 Z$.[1] It is easy to see that the relation

$$\sum_{m=0}^{n_+} a_m F_m Q = \bar{\Delta}^2 \bar{\Delta}^1 Z \tag{3.26}$$

implies

$$\sum_{m=0}^{n_+} a_m F_m = \bar{\Delta}^2_B \bar{\Delta}^1_B Z_B. \tag{3.27}$$

First of all, let us show the relation

$$c Q_1 = \bar{\Delta}^2_B \bar{\Delta}^1_B Z_B, \quad c = \mathrm{const}, \tag{3.28}$$

to imply $c = 0$. Let us write down explicitly the operator $\bar{\Delta}^2_B \bar{\Delta}^1_B$ taken at $u^*_{i2} = 0$:

$$\bar{\Delta}^2_B \bar{\Delta}^1_B Z_B \big|_{u^*_{i2}=0} = \frac{\partial}{\partial u^*_{i1}} \left(u^*_{i1} \frac{\partial}{\partial \bar{u}_j} \frac{\partial}{\partial u^j} - \frac{\partial}{\partial u^*_{j2}} \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} \right) Z_B \Bigg|_{u^*_{i2}=0}. \tag{3.29}$$

Thus we see that r.h.s. of the relation (3.28) for $u^*_{i2} = 0$ can not contain complete product of the variables u^*_{i1}, which fact, in its own turn, implies $c = 0$.

Next, let us introduce the operator

$$L = u^*_{i1} \frac{\partial}{\partial u^*_{i2}}. \tag{3.30}$$

We shall make use of the following its properties

$$\begin{aligned} &[L, \bar{\Delta}^2_B \bar{\Delta}^1_B] = 0, \\ &L^m F_m = (-1)^{m(n_+ - 1)} m! Q_1, \\ &L^m F_k = 0, \quad k < m. \end{aligned} \tag{3.31}$$

By applying the operator L^{n_+} to the relation (3.27), we get

$$n_+! a_{n_+} Q_1 = \bar{\Delta}^2_B \bar{\Delta}^1_B (L^{n_+} Z_B), \tag{3.32}$$

[1] Of course, one can represent the functions $F_m Q$ in the form

$$F_m Q = \bar{\Delta}^1 Z^{(1)}_m = \bar{\Delta}^2 Z^{(2)}, \quad 0 < m < n,$$
$$F_0 Q = \bar{\Delta}^1 Z^{(1)}_0, \quad F_n Q = \bar{\Delta}^2 Z^{(2)}_n.$$

so that $a_{n_+} = 0$. In the same way, applying the operator L^{n_+-1} to the relation (3.27), we get $a_{n_+-1} = 0$, and so on. Finally, we conclude that the relation (3.27) can be fulfilled iff

$$a_m = 0, \quad \forall m. \tag{3.33}$$

So, the general solution to the equation (3.4) is given by the expression (3.25), where the second term in r.h.s. of (3.25) represents "nontrivial common cohomologies" of the operators $\bar{\Delta}^a$.

In physical applications, the variables Φ are split into the subsets $\Phi = (\varphi, c^a, B)$, so that

$$\begin{aligned}
&u = (\varphi_u, c^a_\xi, B_u), \quad \xi = (\varphi_\xi, c^a_u, B_\xi),\\
&\bar{u} = (\bar{\varphi}_u, \bar{c}^a_\xi, \bar{B}_u), \quad \bar{\xi} = (\bar{\varphi}_\xi, \bar{c}^a_u, \bar{B}_\xi),\\
&u^*_a = (\varphi^*_{ua}, c^*_{\xi b|a}, B^*_{ua}), \quad \xi^*_a = (\varphi^*_{\xi a}, c^*_{ub|a}, B^*_{\xi a}),\\
&Q = \prod \varphi_\xi \prod c^a_u \prod B_\xi,\\
&Q_1 = \prod \varphi^*_{u1} \prod c^*_{\xi b|1} \prod B^*_{u1}, \quad Q_2 = \prod \varphi^*_{u2} \prod c^*_{\xi b|2} \prod B^*_{u2}.
\end{aligned} \tag{3.34}$$

The following new ghost number (ngh) values are assigned to all the variables

$$\begin{aligned}
&\mathrm{ngh}(\varphi) = 0, \quad \mathrm{ngh}(c^a) = 1, \quad \mathrm{ngh}(B) = 2,\\
&\mathrm{ngh}(\Phi^*) = -\mathrm{ngh}(\Phi) - 1, \quad \mathrm{ngh}(\bar{\Phi}) = -\mathrm{ngh}(\Phi) - 2.
\end{aligned} \tag{3.35}$$

Then

$$\mathrm{ngh}(F_m Q) = \mathrm{ngh}(Q_1 Q) = -n_{\varphi_u} - 4n_{c^1_\xi} - 3n_{B_u} + 2n_{c^1_u} + 2n_{B_\xi} = -n_{\varphi_u} - n_{B_u} - 2n_{B_\xi} < 0, \tag{3.36}$$

where we have taken into account that $n_{c^1_u} = n_{B_u}$, $n_{c^1_\xi} = n_{B_\xi}$. Thus, in the zero new ghost number sector, the equation (3.4) has only trivial solutions

$$X = \bar{\Delta}^2 \bar{\Delta}^1 Y, \quad \mathrm{ngh}(X) = 0. \tag{3.37}$$

4 Equation $\Delta e^{\frac{i}{\hbar}S} = 0$

.

In this Section we consider the quantum master equation

$$\Delta e^{\frac{i}{\hbar}S} = 0, \quad \varepsilon(S) = 0, \tag{4.1}$$

with the operator Δ defined by the formula (2.1). Of course, if one solves this equation for the exponential in terms of formal power series expansions, then the general solution is given by the formula (2.5), $X = \exp(iS/\hbar)$. If, however, one requires the function S itself to expand in formal power series not only with respect to the variables, but also with respect to the Planck constant $\hbar$, under the extra condition that the classical limit of S at $\hbar \to 0$ does not vanish, then the general solution appears to be of quite different structure. Let us rewrite the equation (4.1) in the following equivalent form

$$\frac{1}{2}(S, S) = i\hbar \Delta S, \tag{4.2}$$

where (F, G) denotes the antibracket

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \Phi^A} \frac{\overrightarrow{\partial}}{\partial \Phi^*_A} G - F \frac{\overleftarrow{\partial}}{\partial \Phi^*_A} \frac{\overrightarrow{\partial}}{\partial \Phi^A} G. \tag{4.3}$$

Let us expand S in formal power series in $\hbar$:

$$S=\sum_{k=0}^{\infty}\hbar^{k}S^{(k)}. \tag{4.4}$$

Then it follows that $S^{(0)}$ satisfies the classical master equation

$$(S^{(0)},S^{(0)})=0. \tag{4.5}$$

Let us solve this equation in terms of formal power series expansions with respect to the variables. Without any lost of generality one can assume that the expansion of $S^{(0)}$ begins with the contribution quadratic in the variables:

$$S^{(0)}=\sum_{k=2}^{\infty}S_k^{(0)},\quad S_k^{(0)}\sim\sum_{l+m=k}O(\Phi^l(\Phi^*)^m). \tag{4.6}$$

In its own turn, the lowest order, $S_2^{(0)}$, also satisfies the master equation

$$(S_2^{(0)},S_2^{(0)})=0. \tag{4.7}$$

Let us denote $\eta=(\omega^A;\omega^*_A)$, $\omega^A=(u^i;-\xi^*_\alpha)$, $\omega^*_A=(u^*_i;\xi^\alpha)$, $\varepsilon(\omega^A)=0$, $\varepsilon(\omega^*_A)=1$, $A=1,\dots,n$. Then the equation (4.7) takes the form

$$\frac{\partial S_2^{(0)}}{\partial\omega^A}\frac{\partial S_2^{(0)}}{\partial\omega^*_A}=0. \tag{4.8}$$

The function $S_2^{(0)}$ has the following general structure

$$S_2^{(0)}=\frac{1}{2}\omega^A L_{AB}\omega^B+\frac{1}{2}\omega^*_A M^{AB}\omega^*_B, \tag{4.9}$$

where L_{AB} is a symmetric matrix, while M^{AB} is an antisymmetric one. Let O_A^B be an orthogonal matrix diagonalizig the matrix L^{AB}:

$$O_A^C O_B^D L_{CD}=\delta_{AB}l_B=\delta_{AB}\zeta_B\lambda_B^2, \tag{4.10}$$

where $\zeta_B=\pm1,0$. Let us perform an anticanonical (AC) transformation (i.e. the one preserving the antibracket (4.3)) such that

$$\omega^A=O_B^A{\Lambda^{-1}}^B_C\omega'^C,\quad \omega^*_A=O_A^B\Lambda_B^C\omega^{*\prime}_C,\quad \Lambda_B^A=\delta_B^A\lambda_B. \tag{4.11}$$

In new variables (we omit primes), the action $S_2^{(0)}$ takes the form

$$S_2^{(0)}=\frac{1}{2}x^i\zeta_i x^i+\frac{1}{2}x^*_i B^{ij}x^*_j+x^*_i C^{i\alpha}y^*_\alpha+\frac{1}{2}y^*_\alpha D^{\alpha\beta}y^*_\beta, \tag{4.12}$$

where $\omega^A=(x^i,y^\alpha)$; variables x^i and y^α correspond to $\zeta_i=\pm1$ and $\zeta_\alpha=0$, respectively. By re-enumerating the variables (which is also an AC transformation), one can place the values of ζ_i to take the standard order

$$\zeta_i=1,\quad i=1,\dots,m_+,\quad \zeta_i=-1,\quad i=m_++1,\dots,m_++m_-. \tag{4.13}$$

Let us substitute the expression (4.12) for $S_2^{(0)}$ into the master equation (4.8):

$$x^i\zeta_i B^{ij}x^*_j+x^i\zeta_i C^{i\alpha}y^*_\alpha=0, \tag{4.14}$$

which yields

$$B^{ij}=0,\quad C^{i\alpha}=0. \tag{4.15}$$

Further, let us restrict ourselves by considering the (physically interesting) case of proper solutions (this means that the Hessian of the function $S_2^{(0)}$ has the rank n). For such solutions, the (antisymmetric) matrix

$D^{\alpha\beta}$ is invertible, and, thus, it can be transformed to take the standard Jordan form, with the help of an orthogonal transformation. Then, by making use of an AC transformation nontrivial only in the y^α, y^*_α sector, the action $S_2^{(0)}$ reduces to take the following canonical form

$$S_{2,c}^{(0)} = \frac{1}{2}x^i\zeta_i x^i + \frac{1}{2}y^*_\alpha\varepsilon^{\alpha\beta}y^*_\beta, \tag{4.16}$$

where $\varepsilon^{\alpha\beta}$ is a matrix whose Jordan form consists of the 2×2–Jordan blocks equal to $i\sigma^2$.

Let us call two functions $F(\eta)$ and $G(\eta)$ AC equivalent, if they are connected with each other by an AC transformation: $G(\eta) = F(\eta'(\eta))$, $\eta \to \eta'$ is AC transformation. Then one can say that the general solution to the master equation (4.8) is AC equivalent to the canonical solution of the form (4.16) characterized by the pair of negative numbers m_+, m_-, $m_+ + m_- \le n$.

Let us turn to solving the master equation (4.5) to higher orders. We suppose that one has already applied the anticanonical transformation reducing the $S_2^{(0)}$ to the canonical form (4.16). Then we get the following equation for $S_3^{(0)}$

$$\begin{aligned} &\gamma S_3^{(0)} = 0, \\ &\gamma = x^i\zeta_i\frac{\partial}{\partial x^*_i} - y^*_\alpha\varepsilon^{\alpha\beta}\frac{\partial}{\partial y^\beta}, \quad \gamma^2 = 0, \end{aligned} \tag{4.17}$$

Introduce the operator d,

$$d = x^*_i\zeta_i\frac{\partial}{\partial x^i} - y^\alpha\varepsilon_{\alpha\beta}\frac{\partial}{\partial y^*_\beta}, \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta^\gamma_\alpha, \tag{4.18}$$

with the properties

$$\begin{aligned} &d^2 = 0, \quad d\gamma + \gamma d = x^i\frac{\partial}{\partial x^i} + y^\alpha\frac{\partial}{\partial y^\alpha} + x^*_i\frac{\partial}{\partial x^*_i} + y^*_\alpha\frac{\partial}{\partial y^*_\alpha} \equiv N, \\ &[N, d] = [N, \gamma] = 0. \end{aligned} \tag{4.19}$$

By making use of the standard reasoning analogous to the one given in Section 2, we find that the general solution to the equation

$$\gamma X = 0 \tag{4.20}$$

has the form

$$X = \gamma Y + c, \quad c = \text{const}. \tag{4.21}$$

Now, returning to the equation (4.17), we conclude that the $S_3^{(0)}$ can be represented in the form

$$S_3^{(0)} = \gamma Y_3 = -(Y_3, S_{2,c}^{(0)}), \quad \varepsilon(Y_3) = 1. \tag{4.22}$$

Let us introduce the function $S'^{(0)}$ AC equivalent to the one $S^{(0)}$:

$$S'^{(0)} = e^{\hat{Y}_3}S^{(0)}, \quad \hat{Y}F \equiv (Y, F). \tag{4.23}$$

Then the $S'^{(0)}$ expands in power series with respect to the variables η, in the form

$$S'^{(0)} = S_{2,c}^{(0)} + S_4'^{(0)} + O(\eta^5), \tag{4.24}$$

and $S_4'^{(0)}$ satisfies the equation

$$\gamma S_4'^{(0)} = 0, \quad S_4'^{(0)} = \gamma Y_4 = -(Y_4, S_{2,c}^{(0)}). \tag{4.25}$$

Continuing this process, we conclude that the general proper solution to the classical master equation (4.5) is AC equivalent to the $S^{(0)}_{2,c}$:

$$S^{(0)} = e^{\hat{Y}^{(0)}} S^{(0)}_{2,c}, \quad \varepsilon(Y^{(0)}) = 1. \tag{4.26}$$

Before we turn to the quantum master equation, let us introduce the operators we call ”gauge transformations” (G transformations)

$$\begin{aligned} &K = e^{[\Delta, Z]_+} = e^{\rho} e^{-\hat{Z}}, \quad \varepsilon(Z) = 1, \\ &\rho = \ln \frac{D(\Theta)}{D(\eta)}, \quad \Theta = e^{-\hat{Z}} \eta. \end{aligned} \tag{4.27}$$

We shall make use of the following properties of G transformations.

Given the function $S(\eta, \hbar)$, let us construct the following $S'(\eta, \hbar)$:

$$e^{\frac{i}{\hbar} S'(\eta,\hbar)} = e^{[\Delta, Z]_+} e^{\frac{i}{\hbar} S(\eta,\hbar)}. \tag{4.28}$$

Then we have

$$\begin{aligned} &S'^{(0)}(\eta) = S'(\eta, 0) = e^{-\hat{Z}^{(0)}} S(\eta, 0) = e^{-\hat{Z}^{(0)}} S^{(0)}(\eta) \\ &\Delta e^{\frac{i}{\hbar} S(\eta,\hbar)} = 0 \quad \Rightarrow \quad \Delta e^{\frac{i}{\hbar} S'(\eta,\hbar)} = 0. \end{aligned} \tag{4.29}$$

Besides, being applied successively, the G transformations result in complete G transformation, again.

To construct a solution to the quantum master equation (4.1), let us apply the following G transformation to the action S:

$$e^{\frac{i}{\hbar} S'} = e^{[\Delta, Y^{(0)}]_+} e^{\frac{i}{\hbar} S}. \tag{4.30}$$

Then $S'^{(0)} = S^{(0)}_{2,c}$, and, by making use of the property $\Delta S^{(0)}_{2,c} = 0$, we find:

$$\gamma S'^{(1)} = 0, \quad S'^{(1)} = \gamma Y^{(1)} + c^{(1)} = -(Y^{(1)}, S^{(0)}_{2,c}) + c^{(1)}. \tag{4.31}$$

Let us introduce the action S'',

$$e^{\frac{i}{\hbar} S''} = e^{-\hbar [\Delta, Y^{(1)}]} e^{\frac{i}{\hbar} S'}. \tag{4.32}$$

whose $\hbar$–power series expansion is

$$S'' = S^{(0)}_{2,c} + \hbar c^{(1)} + \hbar^2 S''^{(2)} + O(\hbar^3). \tag{4.33}$$

It follows from the quantum master equation (4.1) that

$$\gamma S''^{(2)} = 0, \quad S''^{(2)} = \gamma Y^{(2)} + c^{(2)} = -(Y^{(2)}, S^{(0)}_{2,c}) + c^{(2)}. \tag{4.34}$$

Continuing this process, we conclude that the general proper solution to the quantum master equation (4.1) is a result of a G transformation applied to the $S^{(0)}_{2,c}$:

$$e^{\frac{i}{\hbar} S} = e^{[\Delta, Y]_+} e^{\frac{i}{\hbar} (S^{(0)}_{2,c} + c)}. \tag{4.35}$$

Would an analogous result be valid in field theory, this would imply physical triviality of all the field–theoretic models. The locality condition is the only one which causes the existence of physically nontrivial nonequivalent theories.

5 Equation $\bar{\Delta}^a_\hbar e^{\frac{i}{\hbar}S} = 0$

.

To begin with, let us make a few preliminary remarks. We suppose that the variables are split into the following subsets

$$\Phi = (\varphi^I, c^{\alpha b}, B^\alpha), \quad \bar{\Phi} = (\bar{\varphi}_I, \bar{c}_{\alpha b}, \bar{B}_\alpha), \quad \Phi^*_a = (\varphi^*_{Ia}, c^*_{\alpha b|a}, B^*_{\alpha a}). \tag{5.1}$$

The new ghost number values are assigned to all the variables, in the same way as in Section 3, eqs (3.35). The group $Sp(2)$ is supposed to act on the space of all variables (in fact, the $Sp(2)$ applies to the indices a, b). The group is realized as a set of linear changes of variables. We assume that the variables φ, B, $\bar{\varphi}$, $\bar{B}$ are $Sp(2)$ scalars, the c^a form doublet representations, the $\bar{c}_a$, φ^*_a, B^*_a form antidoublet representations, and the $c^*_{b|a}$ transform as a product of two antidoublets.

The operators $\bar{\Delta}^a_\hbar$ have the form

$$\bar{\Delta}^a_\hbar = (-1)^{\varepsilon(\Phi)} \frac{\partial}{\partial \Phi} \frac{\partial}{\partial \Phi^*_a} + \frac{i}{\hbar} \varepsilon^{ac} \Phi^*_c \frac{\partial}{\partial \bar{\Phi}}, \quad \mathrm{ngh}(\bar{\Delta}^a_\hbar) = 1. \tag{5.2}$$

They are nilpotent

$$\bar{\Delta}^a_\hbar \bar{\Delta}^b_\hbar + \bar{\Delta}^b_\hbar \bar{\Delta}^a_\hbar = 0 \tag{5.3}$$

and form an $Sp(2)$ doublet.

We will study the general solution to the equation

$$\bar{\Delta}^a_\hbar e^{\frac{i}{\hbar}S} = 0, \tag{5.4}$$

which appears naturally in $Sp(2)$ symmetric formulation of quantized gauge theories, and which we will call $Sp(2)$ quantum master equation ($Sp(2)$ QME).

Let us introduce the class of operators which we call gauge (G) transformations

$$K = e^{\frac{1}{2}\varepsilon_{ab}[\bar{\Delta}^b_\hbar, [\bar{\Delta}^a_\hbar, F]]_+}, \tag{5.5}$$

where F is an arbitrary function or differential operator. If F is an $Sp(2)$ scalar, then K does the same. If $\mathrm{ngh}(F) = -2$, then $\mathrm{ngh}(K) = 0$ (in this case we say that the operator K preserves ngh). If S satisfies $Sp(2)$ QME (5.4), then S' defined by the formula

$$e^{\frac{i}{\hbar}S'} = K e^{\frac{i}{\hbar}S}, \tag{5.6}$$

does the same. It is easy to check that G transformations applied successively result, again, in a complete G transformation. Besides, as it was shown in [13], an arbitrary change of variables, $\Phi \to \Phi' = F(\Phi)$, can be extended to become a G transformation, in the following sense. Let us represent the $F(\Phi)$ in the form

$$F^A(\Phi) = e^{f^C(\Phi)\frac{\partial}{\partial \Phi^C}} \Phi^A, \tag{5.7}$$

with some functions $f^C(\Phi)$ (such a representation is always possible perturbatively with respect to the variables). Let us consider a G transformation of the form

$$K_f = e^{\frac{i\hbar}{2}\varepsilon_{ab}[\bar{\Delta}^b_\hbar, [\bar{\Delta}^a_\hbar, \bar{\Phi}_A f^A]]_+}. \tag{5.8}$$

Then the relation holds

$$K_f e^{\frac{i}{\hbar}S(\Phi, \Phi^*_a, \bar{\Phi})}\Big|_{\Phi^*_a = \bar{\Phi} = 0} = e^{\frac{i}{\hbar}[S(F(\Phi),0,0) + O(\hbar)]}. \tag{5.9}$$

Moreover, if one deals with the linear change of variables

$$f^C(\Phi) = f^C_D \Phi^D, \tag{5.10}$$

then the operator K_f also reduces, in essential, to the linear transformation

$$S' = e^{U_f} S + \hbar(-1)^{\varepsilon_A} f^A_A, \tag{5.11}$$
$$U_f = f^C_D \Phi^D \frac{\partial}{\partial \Phi^C} - f^D_C \bar{\Phi}_D \frac{\partial}{\partial \bar{\Phi}_C} - f^D_C \Phi^*_{Da} \frac{\partial}{\partial \Phi^*_{Ca}}.$$

If the change $\Phi \to \Phi'$ is $Sp(2)$ covariant and preserves ngh, then the operator K_f does the same.

Now, let us turn to solving of the equation (5.4). We assume that the S is a Boson, $\varepsilon(S) = 0$, preserves ngh: $\mathrm{ngh}(S) = 0$, and is an $Sp(2)$ scalar. Let us seek for a solution in terms of power series expansion with respect to the variables and Planck constant $\hbar$ as well:

$$S = \sum_{k=0}^{\infty} \hbar^k S^{(k)}. \tag{5.12}$$

The $S^{(0)}$ satisfies the $Sp(2)$ master equation (Sp(2) ME)

$$\frac{1}{2}(S^{(0)}, S^{(0)})^a + \varepsilon^{ac} \Phi^*_{Ac} \frac{\partial}{\partial \bar{\Phi}_A} S^{(0)} = 0, \tag{5.13}$$
$$(F, G)^a \equiv F \frac{\overleftarrow{\partial}}{\partial \Phi^A} \frac{\overrightarrow{\partial}}{\partial \Phi^*_{Aa}} G - F \frac{\overleftarrow{\partial}}{\partial \Phi^*_{Aa}} \frac{\overrightarrow{\partial}}{\partial \Phi^A} G.$$

Let us expand the $S^{(0)}$ in power series with respect to the variables

$$S^{(0)} = \sum_{k=2}^{\infty} S^{(0)}_k, \quad S^{(0)}_k \sim \Gamma^k, \tag{5.14}$$

where Γ denotes complete set of variables. The $S^{(0)}_2$ itself satisfies $Sp(2)$ ME

$$\frac{1}{2}(S^{(0)}_2, S^{(0)}_2)^a + \varepsilon^{ac} \Phi^*_{Ac} \frac{\partial}{\partial \bar{\Phi}_A} S^{(0)}_2 = 0. \tag{5.15}$$

The general structure of S^0_2, which preserves ngh and is an $Sp(2)$ scalar, is given by the formula

$$S^{(0)}_2 = \varphi^I D_{1II'} \varphi^{I'} + \varphi^*_{Ia} D^I_{2\alpha} c^{\alpha a} + \bar{\varphi}_I D^I_{3\alpha} B^\alpha + \varepsilon^{ab} c^*_{\alpha b|a} D^\alpha_{4\beta} B^\beta. \tag{5.16}$$

With the help of a linear transformation of the variables φ , in the same way as described in Section 4, one can reduce the matrix D_1 to take the standard form

$$\varphi^I D_{1II'} \varphi^{I'} \quad \to \quad x^i \Lambda^{ij} x^j, \tag{5.17}$$

where Λ_{ij} is diagonal in the sector of Bosonic variables, with m_+ eigenvalues $+1$ and m_- eigenvalues -1, and has the Jordan form in the sector of Fermionic variables, with m Jordan blocks $i\sigma^2$, $\varphi^I = (x^i, y^\mu)$, y^μ are variables complement to x^i (the kernel of the matrix D_1). As a linear transformation of the variables φ is extendable to become a linear transformation of all variables, which is, in turn, a G transformation preserving $Sp(2)$ and ngh, we can suppose that we have it already applied, so that the $S^{(0)}_2$ takes the form

$$S^{(0)}_2 = \frac{1}{2} x^i \Lambda_{ij} x^j + x^*_{ia} D^i_{1\alpha} c^{\alpha a} + y^*_{\mu a} D^\mu_{2\alpha} c^{\alpha a} + \bar{x}_i D^i_{3\alpha} B^\alpha + \bar{y}_\mu D^\mu_{4\alpha} B^\alpha - \varepsilon^{ab} c^*_{\alpha a|b} D^\alpha_{5\beta} B^\beta. \tag{5.18}$$

By substituting the expression (5.18) into the $Sp(2)$ ME (5.15), we get

$$D^j_{1\alpha} = 0, \quad D^\mu_{4\alpha} = D^\mu_{2\beta} D^\beta_{5\alpha}. \tag{5.19}$$

Note that the action S ($S^{(0)}$) for $\bar{\varphi} = \varphi^*_2 = 0$ satisfies the quantum master equation (master equation) in the variables φ, φ^*_1, while for $\bar{\varphi} = \varphi^*_1 = 0$ the actions do the same in the variables φ, φ^*_2. In both cases we

suppose these solutions to be proper, which means that $D^{\mu}_{2\alpha}$ and $D^{\mu}_{4\alpha}$ are square invertible matrices. The $S_2^{(0)}$ is of the form

$$S_2^{(0)} = \frac{1}{2}x^i\Lambda_{ij}x^j + y^*_{\alpha a}D^{\alpha}_{2\beta}c^{\beta a} + \bar{y}_\alpha D^{\alpha}_{4\beta}B^\beta + \varepsilon^{ab}c^*_{\alpha a|b}D^{-1\alpha}_{2\gamma}D^{\gamma}_{4\beta}B^\beta. \tag{5.20}$$

Let us transform the action S by applying the G transformation corresponding to the following linear change

$$c^{\alpha a} \quad\to\quad c'^{\alpha a} = D^{\alpha}_{2\beta}c^{\beta a}, \quad B^\alpha \quad\to\quad B'^\alpha = D^{\alpha}_{4\beta}B^\beta, \tag{5.21}$$

which reduces the action $S_2^{(0)}$ to take the canonical form

$$S_{2,c}^{(0)} = \frac{1}{2}x^i\Lambda_{ij}x^j + y^*_{\alpha a}c^{\alpha a} + \bar{y}_\alpha B^\alpha + \varepsilon^{ab}c^*_{\alpha a|b}B^\alpha. \tag{5.22}$$

The contribution $S_3^{(0)}$ satisfies the equation

$$\begin{aligned}
&(S_2^{(0)}, S_3^{(0)})^a \equiv \gamma^a S_3^{(0)} = 0, \\
&\gamma^a = x^i\Lambda_{ij}\frac{\partial}{\partial x^*_{ja}} - (-1)^{\varepsilon(c^{\alpha a})}c^{\alpha a}\frac{\partial}{\partial y^\alpha} + y^*_{\alpha b}\frac{\partial}{\partial c^*_{\alpha b|a}} + (-1)^{\varepsilon(B^\alpha)}\varepsilon^{ab}B^\alpha\frac{\partial}{\partial c^{\alpha b}} + \\
&\qquad(\bar{y}_\alpha + \varepsilon^{bc}c^*_{\alpha b|c})\frac{\partial}{\partial B^*_{\alpha a}} + \varepsilon^{ab}(x^*_{ib}\frac{\partial}{\partial \bar{x}_i} + y^*_{\alpha b}\frac{\partial}{\partial \bar{y}_\alpha} + c^*_{\alpha c|b}\frac{\partial}{\partial \bar{c}_{\alpha c}} + B^*_{\alpha b}\frac{\partial}{\partial \bar{B}_\alpha}), \\
&\gamma^a\gamma^b + \gamma^b\gamma^a = 0.
\end{aligned} \tag{5.23}$$

To construct a solution to the equation (5.23), let us introduce the operator–valued antidoublet d_a, $\mathrm{ngh}(d_a) = -1$,

$$d_a = x^*_{ia}\Lambda^{-1ij}\frac{\partial}{\partial x^j} - (-1)^{\varepsilon(c^{\alpha a})}y^\alpha\frac{\partial}{\partial c^{\alpha a}} - (-1)^{\varepsilon(B^\alpha)}\varepsilon_{ab}c^{\alpha b}\frac{\partial}{\partial B^\alpha} + \varepsilon_{ab}\bar{x}_i\frac{\partial}{\partial x^*_{ib}} \tag{5.24}$$

with the properties

$$\begin{aligned}
&d_a d_b + d_b d_a = 0, \quad d_a\gamma^b + \gamma^b d_a = \delta^b_a N, \\
&[N, d_a] = [N, \gamma^a] = 0, \\
&N = x^i\frac{\partial}{\partial x^i} + y^\alpha\frac{\partial}{\partial y^\alpha} + c^{\alpha a}\frac{\partial}{\partial c^{\alpha a}} + B^\alpha\frac{\partial}{\partial B^\alpha} + x^*_{ia}\frac{\partial}{\partial x^*_{ia}} + \bar{x}_i\frac{\partial}{\partial \bar{x}_i},
\end{aligned} \tag{5.25}$$

where the operator N is an $Sp(2)$ scalar.

The general solution to the equation

$$\gamma^a X = 0, \quad \mathrm{ngh}(X) = 0, \tag{5.26}$$

has the form

$$X = \frac{1}{2}\varepsilon_{ab}\gamma^b\gamma^a Y + c, \quad c = \mathrm{const}, \quad \mathrm{ngh}(Y) = -2, \tag{5.27}$$

due to the relation

$$N^2 X = \frac{1}{4}\varepsilon_{ab}\gamma^b\gamma^a\varepsilon^{cd}d_d d_c X. \tag{5.28}$$

Note that, being X an $Sp(2)$ scalar, one can choose Y to do the same, as it follows from (5.28).

Thus we have

$$S_3^{(0)} = \frac{1}{2}\varepsilon_{ab}\gamma^b\gamma^a Y_3 = \frac{1}{2}\varepsilon_{ab}(S_2^{(0)}, (S_2^{(0)}, Y_3)^a)^b, \quad \mathrm{ngh}(Y_3) = -2, \tag{5.29}$$

the Y_3 is $Sp(2)$ scalar. Let us apply the following G transformation to S

$$e^{\frac{i}{\hbar}S'} = e^{-\frac{i\hbar}{2}\varepsilon_{ab}[\bar{\Delta}^b,[\bar{\Delta}^a,Y_3]]_+}e^{\frac{i}{\hbar}S}. \tag{5.30}$$

Then the $S'^{(0)}$ expands as

$$\begin{aligned} &S'^{(0)} = S^{(0)}_{2,c} + S^{(0)}_4 + O(\Gamma^5), \\ &\gamma^a S^{(0)}_4 = 0, \end{aligned} \tag{5.31}$$

and so on. Thus, as a result of having applied a series of successive G transformations, the general solution S takes the form

$$\begin{aligned} &S = S^{(0)}_{2,c} + \hbar S^{(1)} + O(\hbar^2), \\ &\gamma^a S^{(1)} = i\Delta^a S^{(0)}_{2,c} = 0, \quad \Delta^a = (-1)^{\varepsilon_A}\frac{\partial}{\partial\Phi^A}\frac{\partial}{\partial\Phi^*_{Aa}}. \end{aligned} \tag{5.32}$$

Continuing this reasoning, we conclude that the general solution to the $Sp(2)$ QME (5.4), preserving $Sp(2)$ and ngh, is G equivalent to the canonical solution

$$e^{\frac{i}{\hbar}S} = e^{\frac{i\hbar}{2}\varepsilon_{ab}[\bar{\Delta}^b,[\bar{\Delta}^a,Y]]_+}e^{\frac{i}{\hbar}(S^{(0)}_{2,c}+c)}, \tag{5.33}$$

where Y is an arbitrary function.

So, one can say again that nontrivial physical result is possible in field theory only because of the locality condition.

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Appendix

In this Appendix we consider briefly how to construct the general solution to the equation

$$\Delta^a_B X_B = 0, \tag{A.1}$$

$$\Delta^a_B = \frac{\partial}{\partial u^i}\frac{\partial}{\partial u^*_{ia}}, \quad \varepsilon(u) = 0, \quad \varepsilon(u^*) = 1, \quad i = 1, \dots, n.$$

In what follows we omit the subscript "B".

As a preliminary step, let us construct the general solution to the equation

$$\Delta^2\Delta^1 U = 0. \tag{A.2}$$

Let us expand U in power series with respect to u^*_{ia}:

$$U = \sum_{m,k=0}^{n} U_{m,k}, \quad U_{m,k} \sim (u^*_1)^m (u^*_2)^k. \tag{A.3}$$

It is obvious that each order contribution $U_{m,n}$ satisfies separately the equation (A.2):

$$\Delta^2\Delta^1 U_{m,k} = 0. \tag{A.4}$$

Let $m = k = n$:

$$U_{n,n} = c(u)Q_1 Q_2, \quad Q_1 \equiv \prod u^*_1, \quad Q_2 \equiv \prod u^*_2. \tag{A.5}$$

The equation (A.4) yields

$$c(u) = c + c_i u^i. \tag{A.6}$$

Now, let m or k (or both of them) be less than n. Let, for example, $k < n$. It follows from the results of Section 2 that

$$\begin{array}{ll} \Delta^1 U_{m,k} = \Delta^2 Y_{m-1,k+1}, & \Delta^2\Delta^1 Y_{m-1,k+1} = 0 \\ \Delta^1 Y_{m-1,k+1} = \Delta^2 Y_{m-2,k+2}, & \Delta^2\Delta^1 Y_{m-2,k+2} = 0 \\ \ldots & \\ \Delta^1 Y_{m-l,k+l} = \Delta^2 Y_{m-l-1,k+l+1}. & \end{array} \tag{A.7}$$

Our further actions depend on which of the two equalities, $m - l - 1 = 0$ or $k + l + 1 = n$, is first satisfied.

Let $m + k \leq n$. Then , for $l = m - 1$ in (A.7), we get

$$\Delta^1 Y_{1,k+m-1} = \Delta^2 Y_{0,k+m}. \tag{A.8}$$

It follows from the results of Section 2 that

$$Y_{0,k+m} = -\Delta^1 Z_{1,k+m}, \tag{A.9}$$

which yields

$$\begin{array}{l} \Delta^1(Y_{1,k+m-1} - \Delta^2 Z_{1,k+m}) = 0, \\ Y_{1,k+m-1} = \Delta^2 Z_{1,k+m} - \Delta^1 Z_{2,k+m-1}, \\ \Delta^1(Y_{2,k+m-2} - \Delta^2 Z_{2,k+m-1}) = 0, \end{array} \tag{A.10}$$

and so on. Finally, we obtain

$$\begin{array}{l} \Delta^1(U_{m,k} - \Delta^2 Z_{m,k+1}) = 0, \\ U_{m,k} = \Delta^2 Z_{m,k+1} + \Delta^1 Z_{m+1,k} \end{array} \tag{A.11}$$

(this representation is also valid for $m = n$, $k = 0$).

Let $m + k > n$. Then, for $l = n - k - 1$ in (A.7), we get

$$\Delta^1 Y_{m+k+1-n,n-1} = \Delta^2 Y_{m+k-n,n}, \quad \Delta^2\Delta^1 Y_{m+k-n,n} = 0, \tag{A.12}$$

which yields

$$\Delta^1 Y_{m+k-n,n} = c(u_1^*)Q_2, \quad c(u_1^*) \sim (u_1^*)^{m+k-1-n}. \tag{A.13}$$

It is convenient to represent the coefficient function $c(u_1^*)$ in the form

$$c(u_1^*) = C_{i_1 \ldots i_{2n-m-k+1}} \frac{\partial}{\partial u_{i_1 1}^*} \cdots \frac{\partial}{\partial u_{i_{2n-m-k+1} 1}^*} Q_1, \tag{A.14}$$

with $C_{i_1\ldots}$ being totally antisymmetric in its indices.

Let us introduce the following notations

$$\begin{array}{l} C_{(m)(k)} \equiv C_{i_1\ldots i_m j_1 \ldots j_k}, \\ (\dfrac{\partial}{\partial u_1^*})^m Q_1 \equiv \dfrac{\partial}{\partial u_{i_1 1}^*} \cdots \dfrac{\partial}{\partial u_{i_m 1}^*} Q_1, \quad (\dfrac{\partial}{\partial u_2^*})^k Q_2 \equiv \dfrac{\partial}{\partial u_{j_1 2}^*} \cdots \dfrac{\partial}{\partial u_{j_k 2}^*} Q_2. \end{array} \tag{A.15}$$

Then

$$c(u_1^*) = C_{(2n-m-k+1)} (\frac{\partial}{\partial u_1^*})^{2n-m-k+1} Q_1. \tag{A.16}$$

The general solution to the equation (A.13) for $Y_{m+k-n,n}$ is

$$Y_{m+k-n,n} = -\Delta^1 Z_{m+k+1-n,n} + u^i C_{i(2n-m-k)}(\frac{\partial}{\partial u_1^*})^{2n-m-k+1} Q_1 Q_2. \tag{A.17}$$

Then we get for $Y_{m+k+1-n,n-1}$

$$\Delta^1(Y_{m+k+1-n,n-1} - \Delta^2 Z_{m+k+1-n,n}) = \xi_1 C_{(2n-m-k)(1)}(\frac{\partial}{\partial u_1^*})^{2n-m-k} Q_1 (\frac{\partial}{\partial u_2^*})^1 Q_2. \tag{A.18}$$

Here $\xi_1 = \pm 1$ is a sign factor which appears when commuting $\partial/\partial u_2^*$ and u_1^*. Its precise value is inessential to us. It follows from (A.18) that

$$Y_{m+k+1-n,n-1} = \Delta^2 Z_{m+k+1-n,n} - \Delta^1 Z_{m+k+2-n,n-1} + \xi_1 u^i C_{i(2n-m-k-1)(1)}(\frac{\partial}{\partial u_1^*})^{2n-m-k-1} Q_1 (\frac{\partial}{\partial u_2^*})^1 Q_2, \tag{A.19}$$

and so on. Finally, we obtain

$$U_{m,k} = \Delta^2 Z_{m,k+1} + \Delta^1 Z_{m+1,k} + u^i C_{i(n-m)(n-k)}(\frac{\partial}{\partial u_1^*})^{n-m} Q_1 (\frac{\partial}{\partial u_2^*})^{n-k} Q_2, \tag{A.20}$$

where all sign factors are absorbed into $C_{(2n-m-k+1)}$.

Thus the general solution to the equation (A.2) can be represented in the form

$$U = \Delta^1 Z_1 + \Delta^2 Z_2 + \sum_{m+k<n} u^i C^{mk}_{i(m)(k)}(\frac{\partial}{\partial u_1^*})^m Q_1 (\frac{\partial}{\partial u_2^*})^k Q_2. \tag{A.21}$$

Now let us return to solving of the equation (A.1). Setting $a = 1$, we find

$$X = \Delta^1 U + c(u_2^*) Q_1. \tag{A.22}$$

The equation (A.1) at $a = 2$ yields

$$\Delta^2 \Delta^1 U = 0. \tag{A.23}$$

Substituting the expression (A.21) for U into (A.22) we obtain

$$X = \Delta^2 \Delta^1 Z + \sum_{m+k<n} C^{mk}_{(m)(k)}(\frac{\partial}{\partial u_1^*})^m Q_1 (\frac{\partial}{\partial u_2^*})^k Q_2. \tag{A.24}$$

Let us show that the second term in r.h.s. of (A.24) cannot be represented in the form $\Delta^2\Delta^1 Z$. Assume that

$$\sum_{m+k<n} C^{mk}_{(m)(k)}(\frac{\partial}{\partial u_1^*})^m Q_1 (\frac{\partial}{\partial u_2^*})^k Q_2 = \Delta^2 \Delta^1 Z \tag{A.25}$$

with some coefficients $C^{mk}_{(m+k)}$. Let m_0 be the maximal of m entering l.h.s. of (A.25). Introduce the operator L,

$$L = u_{i1}^* \frac{\partial}{\partial u_{i2}^*}, \tag{A.26}$$

with the properties

$$\begin{aligned} &[L, \Delta^2 \Delta^1] = 0, \\ &L C^{mk}_{(m)(k)}(\frac{\partial}{\partial u_1^*})^m Q_1 (\frac{\partial}{\partial u_2^*})^k Q_2 = (-1)^{n-1} m C^{mk}_{(m-1)(k+1)}(\frac{\partial}{\partial u_1^*})^{m-1} Q_1 (\frac{\partial}{\partial u_2^*})^{k+1} Q_2. \end{aligned} \tag{A.27}$$

By applying the operator L^{m_0} to the relation (A.25) we get

$$\sum_{m_0+k<n} C^{m_0 k}_{(m_0+k)} Q_1 (\frac{\partial}{\partial u_2^*})^{m_0+k} Q_2 = \frac{(-1)^{m_0(n-1)}}{m_0!} \Delta^2 \Delta^1 (L^{m_0} Z), \tag{A.28}$$

which implies $C^{m_0 k}_{(m_0+k)} = 0$ for all k. Thus we conclude that the relation (A.25) can be fulfilled iff $C^{mk}_{(m+k)} = 0$ for all m, k.

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